A Statistical Decision Method for Economic Evaluation of the Exterior Envelopes of Buildings

Anna Maria Ercules

ASHRAE Associate Member

ABSTRACT

The author describes a multi-stage adaptive statistical decision method which can help to define standards for the insulation of the exterior envelopes of buildings.

Two types of uncertainty are explored.

The first type involves possible changes in economical and competitive factors likely to affect cash flows. These factors will be referred to as "states of the world".

The second type involves the uncertainty of cash flows, once the "states of the world" are known.

The method, based on Bayesian theory, is primarily concerned with the first uncertainty type.

The decision maker is allowed to take differences in the time of future earnings into account, by the use of a discount rate. His attitudes towards risk and non-monetary utilities associated with the possible outcomes are considered in the decision procedure.

As an example it is shown how, in a real-world case, the proposed method can lead to results that are substantially better than the ones obtained by more conventional techniques.

INTRODUCTION

The basic aspects involved in the optimization of building structures from the energy consumption point of view are two:

a. The use of new building typologies or construction techniques aimed to saving energy.

b. The increase in the envelope insulation disregarding construction techniques and building typologies.

The first aspect is strictly related to applied research and its practical large scale application involves long range planning.

The second leads to the definition of standards and it has an immediate influence on all builders' and designers' everyday activity. It does not prevent, of course, the use of new techniques in construction and design.

This note is concerned with the second approach; it describes a means of evaluating the optimal insulation in building envelopes for both the retrofit and the new construction case. The following data must be known: the insulation in existing buildings for the retrofit case, the state of art in building and

---

Anna Maria Ercules, ASHRAE Associate Member, Project Engineer, Mechanical Plant Section, FIAT Engineering Torino (Italy)

design for the new construction case, fuel and insulation costs, meteorological data.

The proposed method is based on statistical decision theory.

PROBLEM DEFINITION

The increase of insulation in an existing building or in a building being planned involves an investiment, i.e. an expense for insulation to be met, hoping for future saving in energy consumption. In general the annual monetary savings, S, obtained by the increase in thermal resistance from the value R to the value b for a unit surface depend on the annual fuel cost savings, C_e, and the annual expense for insulation, C_i. It is:

$$S = C_e - C_i \tag{1}$$

where:

$$C_i = (C_o + q(b-R)) \cdot D \tag{2}$$

$$C_e = g\,(b, R) \cdot L_e \tag{3}$$

being:

C_o : the unit installation cost for insulation
L_e : the energy unit cost.

The function $q\,(b-R)$ shows insulation cost increase as a function of thermal resistance increase. The function $g\,(b, R)$ takes into account climatic data and all the factors relevant to thermal balance that are dependent on thermal resistance variations. The uniform capital recovery factor, D, must be determined according to the law enforced in the country being taken into consideration.

The ratio:

$$P = L_e \cdot g\,(b,R) \;/\; (C_i/D - C_o)$$

will be used in the following pages; it can be considered as the parameter representing insulation and fuel relative marginal cost.

To evaluate the global savings defined by Eq. 1, with reference to the total envelope population and not to a unit surface, the "existing envelope thermal resistance distribution" must be determined. For the purpose of standards determination in the retrofit case and in the new building case, the "existing envelope thermal resistance distribution" has two different meanings. When the matter in question is a retrofit problem, a function $f(R)$ must be determined, showing the relation between the existing surface area and the corresponding thermal resistance R . When the problem in hand relates to the new construction case, the function indicates the probability for a unit of surface of resistance R being built, and it depends on construction and design techniques being adopted. In both cases we will refer to $f(R)$ as a probability density function. Eq. 1, considering the envelope resistance probability distribution, can be written as:

$$S = \int_R g\,(b, R) \cdot L_e - q\,(b-R) \cdot D - C_o D \quad f(R)\, dR \tag{4}$$

$$= P h_1(b) + h_2(b) \tag{5}$$

The upper limit of integration is b , as only the envelopes having a resistance lower than the admissible limit b are relevant to the determination of the benefit arising from the proposed standard.

Summarising: for a given locality, defined by its meteorological data, and for a given envelope resistance density function, the savings that can be achieved enforcing a lower admissible value b for all envelope resistances is a function of the limit b and of the price ratio P .

It is:

$$S = S\,(b, P) \tag{6}$$

P varies with time and is not known with certainty. The decision-maker wishes to choose the optimal b .

THE UTILITY FUNCTION

Annual savings should not be the only factor taken into consideration as there are many equally important determining factors which should be kept in mind. In general the solution requires the maximization of an appropriate function $U = U(b, P)$. The function U will be defined according to the technical or economic factors that are to be taken into account. The simplest utility function is, of course $U(b, P) = S$. Other examples are discussed in this section.

- If comfort is to be considered, a suitable utility function is:

$$U(b, P) = S(b, P) \cdot (1 - e^{-bk}) \tag{7}$$

where k is a constant depending on the meteorological data and the factor $(1 - e^{-bk})$, that grows exponentially from 0 to 1 as b grows from 0 to ∞, represents the comfort related with the envelope resistance b. This function has been obtained considering that room comfort depends on both air and wall temperature, and that wall temperature is a function of wall resistance.

- Walls having a resistance lower than the limit $\hat{b}$ will give origin to condensation on the inside surfaces. To take into account the utility arising from the disappearance of condensation, the following equation can be selected:

$$U(b, P) = S(b, P) \cdot k(b - \hat{b}) \tag{8}$$

where:

$$k(b - \hat{b}) = 1 \quad \text{for} \quad (b - \hat{b}) < 0$$
$$k(b - \hat{b}) > 1 \quad \text{for} \quad (b - \hat{b}) > 0$$

- If decision-maker's risk aversion is to be taken into account, it is:

$$U(b, P) = S(b, P) \cdot \frac{1}{1 + k\, h_s^{-\frac{1}{2}}} \tag{9}$$

where:

h_s is the precision for savings, and k is a constant representing the decision-maker's risk aversion. Remembering Eq. 5, it is:

$$h_s^{-1} = h^{-1} h_1^2(b) + j^2 \tag{10}$$

Savings variance is calculated as the sum of two terms. The first rises from the uncertainty concerning the ratio of insulation and energy costs; the second, called j^2, rises from the uncertainty by which the techniques and the materials used to increase the envelope resistance to the limit b can be forecasted. The value of j^2 can be determined identifying the single uncertainty sources such as insulation material duration, installation difficulty, difference between selected material cost and general average insulation cost, and evaluating the values these factors have for every material type. Factors variance $1/h_i$ and covariance $1/h_{ij}$ between factors can be determined, based on the range of probability that insulation materials have to be selected.

If the effects of the single uncertainty causes are of the additive type, i.e. they sum to each other to give the overall result, it is:

$$j^2 = \sum_i \frac{1}{h_i} + 2 \cdot \sum_{i<j} \frac{1}{h_{ij}} \tag{11}$$

Many relevant factors can be considered by means of the utility function, combining the equations described above.

In the time in which the savings are achieved is relevant, it is:

$$U = \sum_{i=1}^{n} U_i \cdot k_i \cdot \frac{1}{(1 + I)^i} \tag{12}$$

where $(1 + I)^i$ is the uniform capital recovery factor and the constants k_i allows the decision-maker to further vary the sum addenda, giving for example greater importance to savings in the immediate years.

THE OPTIMIZATION PROCEDURE

Utility $U = U(b, P)$ can be represented by a family of curves $U = U(P)$, each curve being characterized by a different value of b (Figure 1).

Let $U^* = U^*(P)$ be the envelope of such curves. It shows the maximum utility that can be achieved for a fixed P. Let b^* be the value of b corresponding to each point in the curve; $b^* = b^*(P)$ is the optimal b for the considered price ratio P.

If P where known with certainty, for example $P = P_0$, $b^* = b^*(P_0)$ would solve the optimization problem.

The solution would be characterized by point A, its coordinates P_0, U_0, and by b_0^* which identifies the curve tangent to U in A.

As a matter of fact only today's value of P is known, and as it can be easily foreseen, it will vary in the future.

In subsequent periods if the decision is not varied, the point A will move on the line identified by $b = b_0$ according to the value of P, and the corresponding utility will not reach the maximum value that can be achieved, shown by $U = U^*$.

Maximum utility can be achieved only varying b, according to P.

In practice this can not be done for many reasons, first of all because of the long time required for the enforcement of standards. On the other hand if a fixed value for b is chosen, based only on today's value of P, a result may be obtained which is very far from the optimal.

In particular standard enforcement, changing demand for both insulation and fuel, will cause P to vary in a way that can not be easily forecasted.

For such reasons it seems to be a reasonable compromise to divide the decision procedure in two steps:

- choice of the optimal lower admissible limit for envelope resistance on the basis of current prices; enforcement by law of this limit;
- situation re-examination after a fixed time period has passed since the law came into force; lowest admissible limit change according to the new conditions.

A mathematical model, described in Appendix A, has been developed to simulate the proposed two step procedure, to calculate its yield and to compare it with more conventional one step techniques. Such a model divides action in two phases:

1. an optimal lower admissible limit for envelope thermal resistance is chosen according to today's value of P. In the following pages the chosen value of b will be called e;
2. based on the results z achieved as a consequence of action e, a better estimate of P will be made, and the final value for b will be chosen. The final value of b will be called s.

The single step procedure, seen as a particular case of the two-step method, can be considered as the optimal procedure when price ratio probability density function does not change with time. In all other cases the two methods will lead to different results.

P is defined as a random variable; the expected value m and precision h, which is defined as the inverted variance, are the parameters characterizing the probability density distribution of P. Also m and h are not known and are considered to be random variables with joint probability density distribution $f(h, m)$.

The expected values of h and m, called h_0 and m_0 characterizing the probability distribution $f(h, m)$, at first are evaluated on the basis of the knowledge currently available. When the first period has elapsed and z has been observed, their evaluation will be improved.

The global utility will be defined as the sum of utilities in period 1 and period 2:

$$U = U_1(e, z) + U_2(s, P) \cdot \frac{1}{1 + I} \qquad (13)$$

where $1+I$ is the discount factor.

RESULTS

An application of the method is described by the example in Appendix B. In that example the new construction case is considered, and degree-days procedure is used for energy consumption valuation. From Figure B.2 it can be seen that for high value of P the curve showing the maximum value of utility approaches a straight line. This means that the higher the fuel cost is in relation to insulation cost, the less influent is the optimization of insulation resistance with respect to future cost variations. Moreover from Figure B.6, it can be seen that for the examined case, if a one-step decision procedure is adopted, i.e. standards are not allowed to change as a consequence of price variation, savings are 20% lower than the savings allowed by a two-step procedure.

DISCUSSION

Reliability in the results obtained by the proposed method depends on input data precision and on the accuracy by which the relation between fuel consumption and envelope thermal resistance can be described. Appendix B simply describes a particular application; the method can be adopted using energy consumption valuation procedures other than degree-days.

The global utility functions for the proposed procedure and for the more conventional one-step procedure show a maximum for the same value of e, as it can be seen in Figure B.6. This can be easily understood considering that U_2^*, expected value of the optimal utility function in period 2 taken with respect to z, is independent of e. The proposed procedure global results can be thought of having been obtained by adding a constant U_2^* to the curve $\bar{U}_1 = \bar{U}_1(e)$, where $\bar{U}_1$ is the expected value of utility in period 1, $U_1(e, z)$, taken with respect to z. Global utility maximum is therefore reached for the value of e which maximizes $\bar{U}_1$.

For the one-step procedure, on the other hand, utility expected value in period 2 is equal to utility expected value in period 1. Global utility is therefore equal to $K \cdot U_1(e)$, where K is a constant taking interest rate into account and it reaches its maximum for the some value of e which maximizes $\bar{U}_1(e)$.

The effect of feed-back caused by the first period decision e on the price ratio have to be studied in detail. In general it can be said that it will lower utility estimate computed in this note, both for the proposed method and the one-step decision.

In the retrofit case special attention must be paid in the determination of the second period lower admissible limit.

Existing envelope distribution has changed as a consequence of period 1 decision, because all resistances not complying to the standard e have been increased to this value. Insulation installed in period 1 will not be removed and consequently period 2 decision can involve only an increase in the admissible limit, i.e. $s \geqslant e$.

This fact will be taken into account in the computations adding the additional constraint:

$$s = e \quad \text{if} \quad s^* < e.$$

Grafically point A (Figure 1) in period 2 for $P < P_0$ will move on the line $b = b_0$ without reaching the optimal curve U^*.

The effect on the final conclusions caused by lack of precision in input data should be studied. However it can be stated that to obtain reliable estimates for savings from the enforcement of a lower limit for building envelopes, the following data and functions must be known with a suitable precision:

- existing (retrofit case), or planned (new construction case) envelope resistance density function;
- meteorological data and methods to compute their effect on energy consumption;
- the function relating thermal resistance with energy consumption;

- the function relating insulation cost and insulation thermal resistance.

Research should be carried out in those fields, to improve knowledge currently available.

CONCLUSIONS

A mathematical model has been described that can be helpful in the evaluation of the economic profitability of the increase of insulation in the external envelope of buildings.

The model can be applied both to the retrofit and the new construction case.

The model can be used with reference both to a single building and to a complex of buildings. In this second case it can help to define standards.

The model does not suggest changes in typology or construction methods, it is strictly related to the evaluation of insulation increases given the "state of the art" of design and the existing insulation envelope situation.

The method leads to:

1. the determination of the lower admissible limits for insulation, which is optimal from the decision maker's point of view; such a limit can be transformed into a standard or a law, when applied to all the existing buildings or to all the buildings being planned for construction in a certain place with given current energy price, insulation costs and meteorological data;
2. the assessment of the economic worth of the decision, keeping in mind that a certain period of time will be required to obtain results of practical significance, and that in the meanwhile price conditions have changed;
3. the assessment of the opportunity to re-examine the chosen decision after a period of time, to take changes of fuel and insulation prices into account.

APPENDIX A

MATHEMATICAL STATEMENT OF THE PROBLEM

A detailed description of the mathematical aspects of the model can be found in 1 .

The following basic data are relevant to the decision problem:

1 - Space of terminal acts: $S = \{s\}$

The decision maker wishes to select a single act s from the domain of potential long-term decisions on the lower limit of admissible envelope resistance.

2 - State space: $Y = \{y\}$

The decision maker believes that the consequence of adopting a terminal act s depends upon some "state of the world" which he can not predict with certainty.

Each potential state which will be labeled by a y with a domain Y, is defined by price ratio in period 2, and is identified by h_o'' and m_o'' .

3 - Family of immediate acts: $E = \{e\}$

To obtain further information on the importance he should attach to each y in Y, the decision maker may select a single short term act e from a family E of potential short term acts, or decisions on the lower limit of admissible envelope resistances.

4 - Sample space: $Z = \{z\}$

The value P assumes as a consequence of the short term decision e will be labelled by a z with domain Z .

5 - Utility evaluation $u(.,.,.,.)$ on $E \times Z \times S \times Y$

The decision maker assigns a utility $u(e,z,s,y)$ to performing a particular e, observing a particular z, taking a particular action s, and then finding what a particular y obtains.

The evaluation of u takes account of the costs or savings (monetary or others) of the short term decision as well as the consequences of the terminal act.

6 - Probability assessment $f_{y,z}(.,../e)$ on $Y \times Z$

For every e in E the decision maker assigns a joint probability measure $f_{y,z}(.,../e)$ or more briefly $f(y,z/e)$ to the cartesian product space $Y \times Z$.

This joint probability measure determines four other probability measures.

a. The marginal measure $f'(y)$ on the state space Y (prior probability). We assume that $f'(y)$ does not depend on e.

b. The conditional measure $f(z/e,y)$ on the sample space Z for given e and y.

c. The marginal measure $f(z/e)$ on the sample space Z for given e but unspecified y.

d. The conditional measure $f''(y/z)$ on the state space for given e and z (posterior probability); the condition e is suppressed because the relevant aspects of e will be expressed as a part of z.

The prime on the measure $f'(y)$ defined in a. indicates that it is the measure the decision maker assigns to Y prior to knowing the outcome z of the immediate action e.

The double prime on the measure $f''(y/z)$ defined in d. indicates that it is the measure on y which he assigns posterior to knowing the outcome z of the experiment.

In taking expectations of random variables, the measure with respect to which the expectation is taken will be indicated either by a subscript appended to the expectation operator E, or by naming the random variable and the conditions in parentheses following the operator. Thus

E'_y or $E'(y)$ is taken with respect to $f'(y)$

$E''_{y/z}$ or $E''(y/z)$ is taken with respect to $f''(y/z)$

$E_{z/ey}$ or $E(z/e,y)$ is taken with respect to $f(z/e,y)$

$E_{z/e}$ or $E(z/e)$ is taken with respect to $f(z/e)$

The decision problem can be stated as:

given E,Z,S,Y and $f(y,z/e)$ how should the decision maker choose an e, and then, having observed z, choose an y in such a way as to maximize his expected utility?

This problem can usefully be represented as a game between the decision maker and a fictitious character we shall call "chance". The game has four moves: the decision maker chooses e, chance chooses z, the decision maker chooses s, and finally chance chooses y. The play is then completed and the decision maker gets the "payoff" $u(e,z,s,y)$.

Although the decision maker has full control over his choice of e and s, he has neither control over, nor perfect knowledge of the choices of z and y which will be made by chance.

We have assumed, however, that he is able in some way or another to assign probabilities measures over these choices, and the moves in the game proceed in accordance with these measures as follows:

Move 1 : The decision maker selects e in E.
Move 2 : Chance selects z in Z according to the measure $f(z/e)$.
Move 3 : The decision maker selects a s in S.
Move 4 : Chance selects a y in Y according to measure $f''(y/z)$.
Payoff : The decision maker receives $u(e,z,s,y)$.

The Decision Tree

When the spaces $E, Z, S,$ and Y are all finite the flow of the game can be represented by a tree diagram; and although a complete diagram can actually be drawn only if the number of elements involved in E, Z, S and Y is very small, even an incomplete representation of the tree can aid our intuition.

A partial tree of this sort is shown in figure A-1, where D denotes the decision maker and C denotes chance.

At move 1 D chooses some branch e of the tree; at move 2 C chooses a branch z; at move 3 D chooses s; at move 4 C chooses y, and finally D receives the payoff $u(e,z,s,y)$.

The problem can be worked out using a form of analysis which proceeds by working backwards from the end of the decision tree (right hand side or figure A-1) to the initial starting point: instead of asking which experiment e the decision maker should choose at move 1 when he knows neither the moves which will subsequently be made by chance, we start by asking which terminal act he should choose at move 3 if he had already performed a particular experiment e and observed a particular outcome z. Even at this point with a known history (e,z) the utilities of the various possible terminal acts are uncertain because the y which will be chosen by chance at move 4 is still unknown; but this difficulty is easily resolved by treating the utility of any s for given (e,z) as a random variable $U(e,z,s,y)$ and applying the operator $E''_{y/z}$ which takes the expected value of $u(e,z,s,y)$ with respect to the conditional measure $f''(y/z)$.

Simbolically we can compute for any given history (e,z) and any terminal act s:

$$U(e,z,s) = E''_{y/z}\, u(e,z,s,y) \tag{A.1}$$

this is the utility of being at the juncture (e,z,s) looking forward, before chance has made a choice of y.

Now, since the decision maker's objective is to maximize his expected utility, he will, if faced with a given history (e,z), choose the s for which $U(e,z,s)$ is greatest; since he is free to make his choice as he pleases, we may say that the utility of being at move 3 with history (e,z) and the choice of s still to make is

$$U^*(e,z) = \max_s U(e,z,s) \tag{A.2}$$

After we have computed $U^*(e,z)$ in this way for all possible histories (e,z), we are ready to attack the problem of the initial choice of an experiment.

At this point, move 1, the utilities of the various possible experiments are uncertain only because the z that will be chosen by chance at move 2 is still unknown, and the difficulty is resolved in exactly the same way the difficulty in choosing s given (e,z) was resolved: by putting a probability measure over chance's moves and taking expected values.

In other words $U^*(e,z)$ is a random variable at move 1 because z is a random variable, and we therefore define for any e

$$U^*(e) = E_{z/e} U^*(e,z) \tag{A.3}$$

where $E_{z/e}$ expects with respect to the marginal measure $f(z/e)$.

Now again the decision maker will wish to choose the e for which $U(e)$ is greatest; and therefore we may say the utility of being at move 1 with the choice of e still to make is

$$U^* = \max_e U^*(e) = \max_e E_{z/e} \max_s E_{y/z} U(e,z,s,y) \tag{A.4}$$

Posterior Probability Calculation: Baye's Theorem

If the prior distribution of the random variable $\tilde{y}$ has a density function f', it follows from Baye's theorem that the posterior distribution of $\tilde{y}$ has a density function f'' whose value at y for the given z is

$$f''(y/z) = f'(y) f(z/y) N(z) \tag{A.5}$$

where $N(z)$ is simply the normalizing constant defined by the condition

$$\int_y f''(y/z)\,dy = N(z) \int_y f'(y) f(z/y)\,dy = 1 \tag{A.6}$$

Calculations are particularly simple if

$$f(z/y) = \frac{1}{\sqrt{2\pi}} h^{\frac{1}{2}} e^{-(z-m)^2 h \frac{1}{2}} \tag{A.7}$$

i.e. $f(z/y)$ is normal with expected value m and precision h.

Both parameters of the normal process are unknown and are to be treated as random variables $\tilde{m}$ $\tilde{h}$; the most convenient joint distribution of the two variables is what we call a normal-gamma distribution, defined by

$$f_{N\gamma}(m,h/m_o,h_o,w_o,n_o) = f_N(m/m_o,n_o h) f_{\gamma 2}(h/h_o,w_o) \tag{A.8}$$

$$\propto e^{-\frac{1}{2}hn_o(m-m_o)^2}(n_o h)^{\frac{1}{2}} \cdot e^{\frac{1}{2}w_o\frac{h}{h_o}} h^{\frac{1}{2}w_o - 1} \tag{A.9}$$

$$\begin{aligned} -\infty &< m < +\infty \\ h &< 0 \\ w_o, m_o, h_o, n_o &< 0 \end{aligned} \tag{A.10}$$

y is identified by (m_o, h_o)

If the prior distribution of (m,h) is normal-gamma with parameters (m_o', n_o', h_o', w_o') and if the experiment yields a z, the posterior distribution of (m,h) will be normal-gamma with parameters:

$$m_o'' = \frac{n_o' m_o' + z}{n_o' + 1} \tag{A.11}$$

$$n_o'' = n_o' + 1 \tag{A.12}$$

$$h_o'' = \frac{w_o' + 1}{\frac{w_o'}{h_o'} + n_o' m_o'^2 + z^2 - n_o'' m_o''^2} \tag{A.13}$$

$$w_o'' = w_o' + 1 \tag{A.14}$$

APPENDIX B

AN APPLICATION

As an example, a case related to the Italian situation will be considered. The following hypotheses will be made:

- Unit cost for insulation thermal resistance is a constant; in other words, this means that the cost to increase insulation does not depend on thickness. In this case it is:

$$q(b-R) = L_i \cdot (b-R) \tag{B.1}$$

  where L_i is a constant.

- For a fixed interior temperature and a given climatic exterior situation, thermal flow through the envelope is inversely proportional to envelope thermal resistance. In this case the function $g(b,R)$, defined in Eq. 3, can be written as:

$$g(b,R) = \frac{C}{R} \cdot \frac{b-R}{b} \tag{B.2}$$

  where:

  $\frac{C}{R}$ represents annual fuel consumption for a unit surface, evaluated for the situation preceding the enforcement of the resistance lower limit.

  C is a constant taking into account climatic conditions, interior temperature, and combustion efficiency; it has been evaluated by the degree-days method.

  $\frac{b-R}{b}$ represents the percentage decrease of thermal flow through the envelope.

- The "existing envelope thermal resistance distribution" is a gamma-1 function (Fig. B.1).

$$f(R/ry) = \frac{e^{-Ry}(y\ R)^{r-1}}{(r-1)} \cdot y \qquad \begin{matrix} R > 0 \\ r,y > 0 \end{matrix} \tag{B.3}$$

- A new construction case is examined.

Input data are listed in Table B.1 . To approach values 1) and 2) in Table B.1, in Eq. B.3, it will be set: $r = 3$, $y = 4$. The savings obtained enforcing a lower admissible limit b for all envelope resistances, dropping a constant multiplier are:

$$S(b,P) = \int_0^b \left[f(R) \cdot \left(\frac{C}{R \cdot b} P - 1\right) \cdot (R-1) \right] dR - K \tag{B.4}$$

$$= P \cdot \left[e^{-by}\left(Cy + \frac{2 \cdot C}{b}\right) + C \cdot y - \frac{2 \cdot C}{b} \right] +$$

$$e^{-by}\left(-b^2 \cdot y - 4b - \frac{6}{y}\right) - 2y + \frac{6}{b} - K \tag{B.5}$$

The following utility function has been chosen:

$$U(b,P) = S(b,P) \cdot (1 - e^{-kb}) \tag{B.6}$$

where k is a constant.

In the calculations and figures utility is measured in arbitrary units, savings are computed making reference to a one-year period, all other variables in S.I. units.

Figure B.2 shows the curves $U = U(P)$ having b as a parameter. As it can be easily seen, the proposed method is usefull particularly for low value of P, where the curve $U^* = U^*(P)$ trend is least approximated by a straight line.

Figure B.3 shows the curve $b^* = b^*(P)$. It represents the decisions to be adopted when P is known with certainty, and the decisions to be adopted in the first period when the price ratio is a random variable.

The used method is logically equivalent to the method described in the Appendix A.

The "state of the world" y is identified by h''_o and m''_o , expected value of precision and expected value of price ratio P in period 2; its density function is determined by means of Baye's theorem. For the calculation purpose it is equivalent to the random variable t .

In Appendix A only global utility was considered; here, to simplify calculations, period 1 and period 2 utilities are computed separately. Period 2 utility is maximized with respect to s , with z as a parameter; global utility is computed as a function of e and z , and at last its expected value, taken with respect to z , is maximized.

Let P be a random variable with normal density function, having expected value m and precision h ; let h,m have a joint probability distribution of normal-gamma type; it is:

$$f(P/h,m) = \frac{1}{\sqrt{2\pi}} h^{\frac{1}{2}} e^{-\frac{1}{2}h(P-m)^2} \tag{B.7}$$

From Eq. B.7 the marginal density function for P in period 2 is:

$$\begin{aligned} f''(t/e,z) &= f''(P/e,z) \\ &= f''(P/m''_o, h''_o) \\ &= \iint_{hm} f(P/m,h)\ f\ (m,h/m''_o,h''_o)\ d\,m\,dh \end{aligned} \tag{B.8}$$

$$\propto \iint_{hm} \frac{h^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{h}{2}(P-m)^2} \frac{(n''_o h)^{\frac{1}{2}}}{\sqrt{2\pi}} \cdot e^{\left[-\frac{n''h}{2}(m-m'')^2 - \frac{h}{2}\frac{w''_o}{h''_o}\right]} h^{\left[\frac{1}{2}w'_o - 1\right]} d\,m\,d\,h$$

Period 2 utility expected value $U_2(s/e,z)$ can be obtained applying operator $E''_{t/z}$. It is:

$$U_2(s/e,z) = E''_{t/z} U_2(s,t/e,z)$$

$$= \int U_2(s,t/e,z) f''(t/e,z) dt$$

$$= (h_1(s)m''_o + h_2(s))(1 - e^{-ks}) \qquad (B.9)$$

$$= (h_1(s)\frac{n'_o m'_o + z}{n'_o + b} + h_2(s))\ (1 - e^{-ks})$$

Functions h_1 and h_2 have been defined in Eq. 5.

For any value of z (i.e. of the price ratio in period 1) a curve can be drawn showing the utility expected value in period 2 as a function of the period 2 decision s. Such a curve reaches its maximum value U_2^* for $s = s^*$. The variables s^* and U_2^* as functions of z are shown in Figure B.4.

The first period utility is:

$$U_1(e,z) = (h_1(e)z + h_2(e)) \cdot (1 - e^{-ke}) \qquad (B.10)$$

Total utility can be computed by:

$$U(e,z) = U_1(e,z) + U_2(z)\frac{1}{1+I} \qquad (B.11)$$

where I is the interest rate.

The curve family $U = U(z/e)$ having e as a parameter is shown in Figure B.5.

The total utility expected value is:

$$\bar{U}(e) = \int U(e,z)\ f(z)\ dz \qquad (B.12)$$

$f(z)$, marginal density function of P in period 1, can be computed as

$$f(z) = f'(P)$$

$$= \iint_{hm} f(P/m,h)\ f(m,h/m'_o, h'_o)\ d\,m\,d\,h \qquad (B.13)$$

$$\propto \iint_{hm} \frac{h^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{h}{2}(P-m)^2} \frac{(n'_o h)^{\frac{1}{2}}}{\sqrt{2\pi}} e^{\left[-\frac{n'_o h}{2}(m-m'_o)^2 - \frac{h}{2}\frac{w'_o}{h'_o}\right]} h^{\left[\frac{1}{2}w'_o - 1\right]} dmdh$$

It is:

$$\bar{U}(e) = \int_Z (U,(e,z) + \frac{U_2(z)}{1+I})\ f(z)\ dz \qquad (B.14)$$

Figure B.6 shows the function $\bar{U} = \bar{U}(e)$.

If the calculations shown in the above procedure are repeated choosing for s the value $s = e$ instead of $s = s^*$, the expected value obtained from B.14, that will be called $\bar{U}' = \bar{U}'(e)$, shows the utility that can be obtained using a one-step decision algorithm without varying in period 2 the lower admissible limit for the envelope resistance. The function $\bar{U}' = \bar{U}'(e)$ is shown in Figure B.6.

The advantages of the proposed procedure can be easily seen. The segment $U - U'$ represents the value of information gathered in period 1.

NOMENCLATURE

C_e : annual fuel cost savings.
Ci : annual expense for insulation.
C_o : unit installation cost for insulation.
D : uniform capital recovery factor.
I : interest rate.
L_e : energy unit cost.

L_i : insulation resistance unit cost.
P : unit fuel cost and unit insulation cost ratio (random variable).
b : lower admissible limit for envelope resistances.
s : value of b in period 2.
z : the value P assumes in period 1 after having taken action e.
t : the value P assumes in period 2 after having taken action s.
R : existing envelope resistance (retrofit case), envelope resistance of buildings that will be designed (new construction case).
S : savings that can be obtained enforcing a lower admissible limit b for all envelope resistances;
e : value of b in period 1.
h : precision for P (random variable). It is defined as the inverse of variance.
h_o : expected value of h.
m : expected value for P (random variable).
m_o : expected value of m.
y : state of the world, defined by (m, h).
U(e,z,s,t) : total utility.
$U_1(e,z)$: utility in period 1.
$U_1(e)$: expected value of utility in period 1.
$U_2(s,t/e,z)$: utility in period 2, given z and e.
$U_2(s/e,z)$: expected value of utility in period 2 taken with respect to t.
$U_2^*(z,e)$: maximum value of the function $U_2(s/e,z)$, it corresponds to the value $s = s^*$.
$\bar{U}(e)$: total utility expected value.
$f(m,h/m_o,h_o)$: m and h joint probability density function.
f(P/m,h) : P probability density function, given m and h.
f(R) : R probability density distribution.
g(b,R) : annual energy consumption reduction when the insulation resistance is increased from R to b.
q(b-R) : direct cost occurred to increase insulation resistance from R to b, for a unit surface (it does not include installation cost).
' : the prime indicates the value the variables parameters and function assume in period 1.
" : the double prime indicates the value the variables parameters and function assume in period 2.

REFERENCES

[1] Raiffa, H. and Schaifer, R., Applied Statistical Decision Theory, Division of Research Harward University School, Boston, 1961.

BIBLIOGRAPHY

Dwass, M., Probability and Statistics, Benjamin Inc., New York 1970.

Magee, J.F., "Decision Tree for Decision Making", Harward Business Review, Boston, july 1964.

Magee, J.F., "How to use decision trees in capital investment", Harward Business Review, Boston, September 1964.

Howard, R.A., "Decision Analysis: Perspectives on Inference, Decision and Experimentation", Proceedings of the IEEE, 58, May 1970.

1977 Fundamentals, ASHRAE Handbook & Product Directory, New York, 1977.

TABLE B.1

Input data for the example application

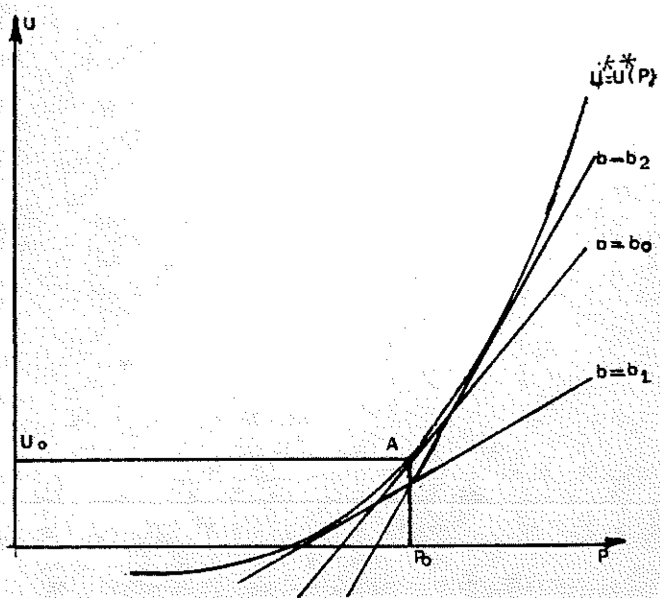
1) Existing envelope resistance distribution average	6. $\frac{m^2\ {}^\circ K}{W}$
2) Existing envelope resistance distribution most likely occurrence	0.43 $\frac{m^2\ {}^\circ K}{W}$
3) Degree-days (base temperature: 19°C)	1500.

4) Insulation resistance unit cost (today's value) $2300. \dfrac{Lit^*}{\dfrac{m^2\ {}^\circ K}{W}} \cdot m^2$

5) Energy unit cost (today's value) $10^{-6}\ \ 6{,}5\ \dfrac{Lit^*}{J}$

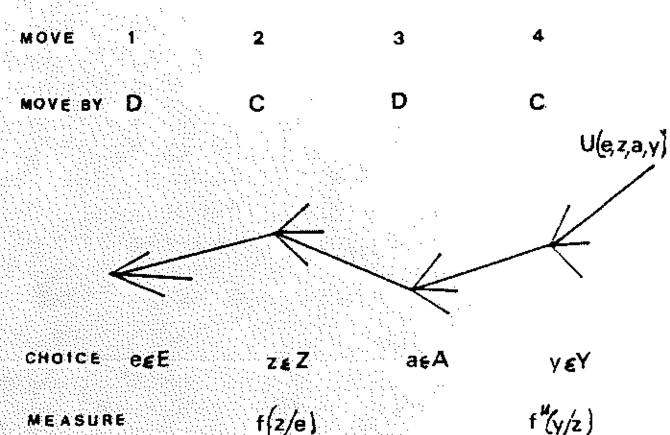
* Italian Lira

Figure 1



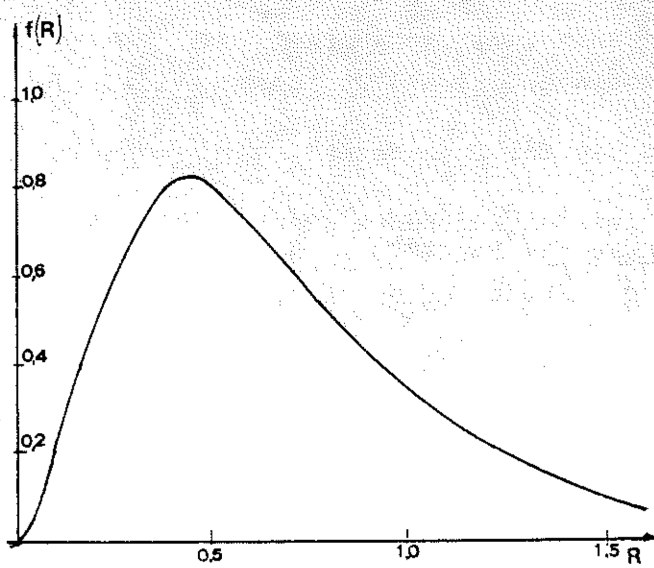
Utility as a function of P, for a given b. Choice of the optimal b.

Figure A.1



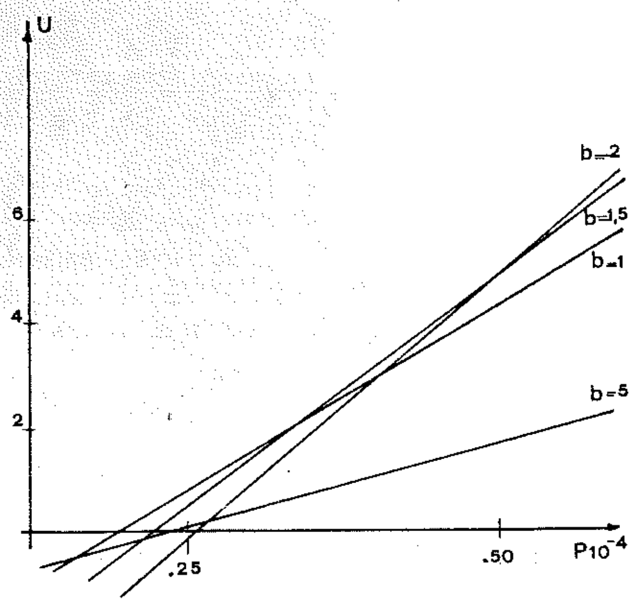
Tree diagram representing the decision procedure.

Figure B.1



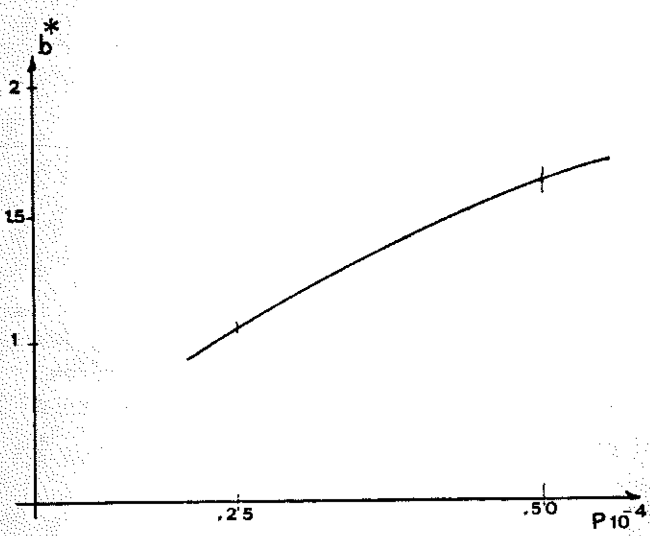
Envelope resistance density function.

Figure B.2



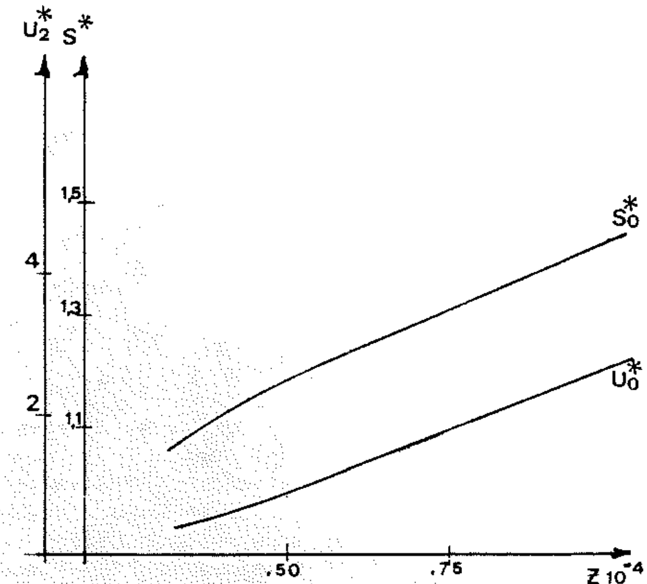
Utility as a function of P, in the example case.

Figure B.3



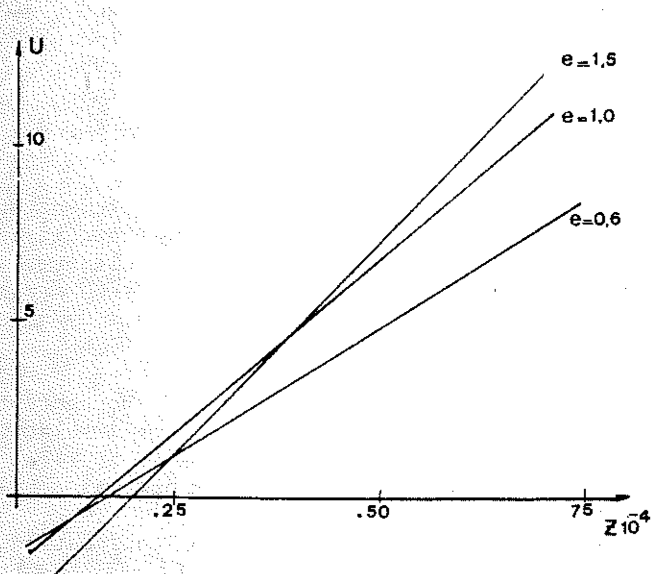
Choice of the optimal b in the proposed example case, considering P as a deterministic variable.

Figure B.4



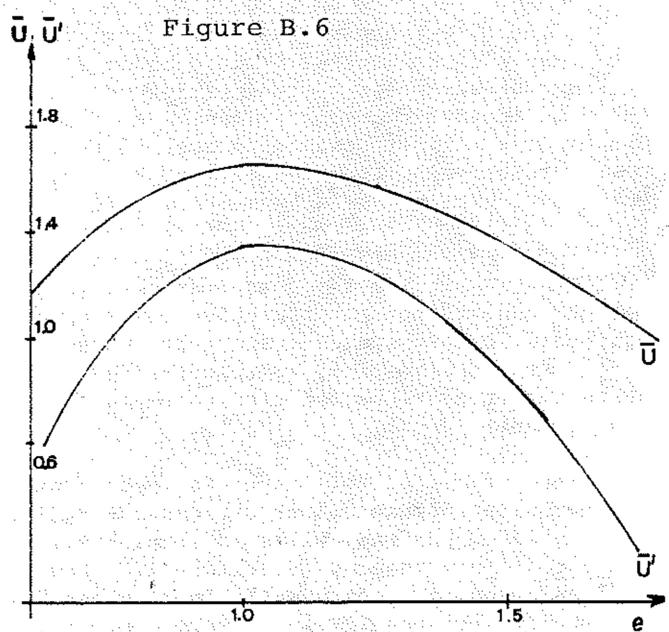
Optimal value for s and corresponding utility in period 2 in the example case.

Figure B.5



Total utility for the proposed example as a function of z having e as a parameter

Figure B.6



Optimal utility for the proposed example adopting a two-step procedure and a single-step procedure.